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Optimal Semicomputable Approximations to Reachable and Invariant Sets

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Abstract. In this paper we consider the computation of reachable, viable and invariant sets for discrete-time systems. We use the framework of type-two effectivity, in which computations are performed by Turing machines with infinite input and output tapes, with the representations of computable topology. We see that the reachable set is lower-semicomputable, and the viability and invariance kernels are upper-semicomputable. We then define an upper-semicomputable over-approximation to the reachable set, and lower-semicomputable under-approximations to the viability and invariance kernels, and show that these approximations are optimal.

1. Introduction

The computation of reachable, viable and invariant sets are important problems in nonlinear systems theory. For safety-critical applications, it is important to be able to compute these sets accurately, taking care to control the error bounds. However, the results of [6] show that the reachable set is lower-semicomputable, but not upper-semicomputable, which means that it is impossible to compute arbitrary accurate upper bounds to the reachable set. Instead, it is possible to upper-semicompute the *chain-reachable set*, which over-approximates the reachable set. These results were extended to viability and invariance kernels in [7], which were shown to be upper-semicomputable, but to have robust under-approximations which are lower-semicomputable.

We consider computability in the framework of *type-two effectivity* developed by Weihrauch [23] and co-workers. In this theory, computations are performed by stan-

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dard Turing machines with *input*, *output* and *work tapes*. Unlike standard computability theory (type-one effectivity) in which inputs and outputs are *words* (elements of Σ^*), type-two machines can compute on *sequences* (elements of Σ^ω). This allows representations of, and computations on, the standard objects of analysis and topology, such as real numbers, open, closed and compact sets, continuous functions and semicontinuous multivalued functions. Computable topology provides a *standard representation* for elements of a topological space, which allows the extraction of approximations by *denotable* elements with various error bounds. The main result of the theory is that only functions and operators which are continuous with respect to the underlying topology are computable in the standard representation. For this paper, we study semicomputable operators, which are continuous with respect to lower or upper topologies, and hence are called semicontinuous.

The purpose of this paper is to consider the approximations to the reachable set and the viability and invariance kernels, and show that they provide the optimal possible computable approximations. More precisely, the main results are to show that the chain-reachable set the optimal upper-semicomputable over-approximation to the reachable set, and that the viability and invariance kernels are the optimal lower-semicomputable under-approximations to the viability and invariance kernels. These results have major implications for tool developers; any tool which computes an over-approximation to the reachable set of a nonlinear system can do no better than approximate the chain-reachable set, and any tool which computes an under-approximation to the viability or invariance kernels can do no better than the robust viability and invariance kernels.

We remark that the negative computability results presented here assume that the *only* information we have about sets and systems are lower and upper approximations. If more detailed information is available (e.g. an algebraic description in terms of polynomials with rational coefficients) then it may be possible to determine these sets exactly, even if they differ. In other words, a lack of computability in the approximative sense used here does not imply a lack of computability in some other computational framework. However, a lack of computability in the approximative sense does indicate that the problem is non-robust, so results obtained using exact methods may not be physically meaningful. The framework of computable analysis can deal with arbitrary (semi)continuous systems, whereas algebraic methods can handle systems which are not semicontinuous, but severely restricts the class of continuous systems which can considered.

There is a large body of literature on approximation methods in viability theory such as that by Aubin and Frankowska [4] and Cardaliaguet et al. [5]. Approximation methods based on ellipsoidal techniques have been considered by Kurzhanski and Varaiya [17], [18]. A number of applications of set-valued methods to control problems are given by Szolnoki [22]. The relation between reachability and chain reachability has been considered by Asarin and Bouajjani [1]. Optimal controllers have been computed by Junge and Osinga [15] using the tool GAIO. An alternative approximation framework based on first-order logic over the reals is given by Fränzle [10], [11].

There are already many tools which compute approximations to the reachable set, such as d/dt [2], CheckMate [21] and HyTech [13] for linear hybrid systems, and HyperTech [14] and PHAVer [12] for over-approximation of reachable sets. Computation of reachable sets can also be performed by the general-purpose package GAIO [9] for set-based computations.

The paper is organised as follows. In Section 2 we review some material on sets and multivalued functions, and show how to construct semicontinuous functions lying in an open set. In Section 3 we review the elements of computable topology we use. The main results are contained in Section 4. We give some conclusions and directions for future research in Section 5.

2. Topologies on Sets and Semicontinuous Maps

We now introduce some basic topology of locally compact Hausdorff spaces, which can be found in [19], and of hyperspaces of open, closed and compact sets.

2.1. *Open*, *Closed and Compact Sets*

We let X be a locally compact, second countable Hausdorff space with topology τ. Let $\mathcal{O}$, $\mathcal{A}$ and $\mathcal{K}$ denote the open, closed and compact subsets of X, respectively.

The space of closed and compact sets can be topologised using the *hit-and-miss* topologies of Fell and Vietoris. An open set in the *lower* topology on $\mathcal{A}$ consists of all sets which "hit" a given open subset of X, and hence can be seen as giving "positive" information about its elements. An open set in the *upper* topology on $\mathcal{A}$ or $\mathcal{K}$ consists of all sets which "miss" a given compact or closed subset of X, and hence can be seen as giving "negative" information about its elements.

Definition 2.1.

1. The lower topology $\tau_<^{\mathcal{O}}$, generated by sets of the form $\{U \in \mathcal{O} \mid C \subset U\}$ for $C \in \mathcal{K}$.
2. The lower topology $\tau_<^{\mathcal{A}}$, generated by sets of the form $\{A \in \mathcal{A} \mid A \cap J \neq \emptyset\}$ for $J \in \mathcal{O}$.
3. The upper Fell topology $\tau_>^{\mathcal{A}}$, generated by sets of the form $\{A \in \mathcal{A} \mid A \cap B = \emptyset\}$ for $B \in \mathcal{K}$.
4. The upper Vietoris topology $\tau_>^{\mathcal{K}}$, generated by sets of the form $\{C \in \mathcal{K} \mid C \cap B = \emptyset\}$ for $B \in \mathcal{A}$.

The Fell topology $\tau^{\mathcal{A}}$ on $\mathcal{A}(X)$ is generated by $\tau_<^{\mathcal{A}}$ and $\tau_>^{\mathcal{A}}$, and the Vietoris topology on $\mathcal{K}$ is generated by the restriction of $\tau_<^{\mathcal{A}}$ and $\tau_>^{\mathcal{K}}$.

If β is a base for the topology on X, we can construct countable bases for $\tau_<^{\mathcal{O}}$, $\tau_<^{\mathcal{A}}$, $\tau_>^{\mathcal{A}}$ and $\tau_>^{\mathcal{K}}$ as follows:

$$
\begin{aligned}
\beta_<^{\mathcal{O}} &:= \{\{U \in \mathcal{O} \mid \bar{I}_j \subset U,\ i = 1, \ldots, k\} \mid I_1, \ldots, I_k \in \beta\},\\
\beta_<^{\mathcal{A}} &:= \{\{A \in \mathcal{A} \mid A \cap J_i \neq 0,\ i = 1, \ldots, k\} \mid J_1, \ldots, J_k \in \beta\},\\
\beta_>^{\mathcal{A}} &:= \{\{A \in \mathcal{A} \mid A \cap \bar{I}_j = \emptyset,\ i = 1, \ldots, k\} \mid I_1, \ldots, I_k \in \beta\},\\
\beta_>^{\mathcal{K}} &:= \{\{C \in \mathcal{K} \mid C \subset J_1 \cup \cdots \cup J_k\} \mid J_1, \ldots, J_k \in \beta\}.
\end{aligned}
\tag{1}
$$

We henceforth use the convention that A, B represent closed sets, C represents a compact set, U, V represent open sets and I, J, K represent basic open sets.

2.2. *Semicontinuous and Continuous Multivalued Functions*

The results of this paper hold for semicontinuous functions on locally compact Hausdorff spaces.

We typically specify a multivalued map $F\colon X \rightrightarrows Y$ by a giving a single-valued map $X \to \mathcal{P}(Y)$. The action of F on sets is then given by $F(A) := \{y \in Y\colon \exists\, x \in A,\ y \in F(x)\}$ for $A \in \mathcal{P}X$. If $F\colon X \rightrightarrows Y$ and $G\colon Y \rightrightarrows Z$, the *composition* of F and G is $G \circ F\colon X \rightrightarrows Z$ given by $G \circ F\,(x) := G(F(x)) = \{z \in Z\colon \exists\, y \in Y,\ y \in F(x) \text{ and } z \in G(y)\}$. If $F, G\colon X \rightrightarrows Y$, we write $F \subset G$ if $F(x) \subset G(x)$ for all x.

There are two natural set-valued preimages of $F\colon X \rightrightarrows Y$: the *weak preimage* $F^{-1}(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}$ and the *strong preimage*, $F^{\Leftarrow}(B) = \{x \in X\colon F(x) \subset B\}$. We say F is *lower-semicontinuous* if $F^{-1}(U)$ is open whenever U is open, or equivalently, if $F^{\Leftarrow}(A)$ is closed whenever A is closed. F is *upper-semicontinuous* if $F^{-1}(A)$ is closed whenever A is closed, or equivalently, if $F^{\Leftarrow}(U)$ is open whenever U is open. A function F is *weakly upper-semicontinuous* if $F^{-1}(C)$ is closed whenever C is compact. A multivalued function is (*weakly*) *continuous* if it is both lower-semicontinuous and (weakly) upper-semicontinuous.

We say a function has *closed values* if $F(x)$ is closed for all x, denoted $F\colon X \to \mathcal{A}(Y)$, and *compact values* if $F(x)$ is compact for all x, denoted $F\colon X \to \mathcal{K}(Y)$.

It is easy to see that a closed-valued function $F\colon X \rightrightarrows Y$ is lower-semicontinuous if, and only if, it is $(\tau^X; \tau_<^{\mathcal{A}(Y)})$-continuous, and a compact-valued function $F\colon X \rightrightarrows Y$ is upper-semicontinuous if, and only if, it is $(\tau^X; \tau_>^{\mathcal{K}(Y)})$-continuous. However, the set of $(\tau^X; \tau_>^{\mathcal{A}(Y)})$-continuous functions consists of only weakly upper-semicontinuous closed-valued functions.

We define $LSC^{\mathcal{O}}(X \rightrightarrows Y)$ to be the set of lower-semicontinuous open-valued functions, $LSC^{\mathcal{A}}(X \rightrightarrows Y)$ to be the set of lower-semicontinuous closed-valued functions, $USC^{\mathcal{A}}(X \rightrightarrows Y)$ to be the set of weakly upper-semicontinuous closed-valued functions, and $USC^{\mathcal{K}}(X \rightrightarrows Y)$ to be the set of upper-semicontinuous compact-valued functions. We denote closed-valued weakly continuous functions by $C^{\mathcal{A}}$ and compact-valued continuous functions by $C^{\mathcal{K}}$.

The topologies $\tau_<^{\mathcal{MO}}$, $\tau_<^{\mathcal{MA}}$, $\tau_>^{\mathcal{MA}}$ and $\tau_>^{\mathcal{MK}}$ on $LSC^{\mathcal{O}}(X \rightrightarrows Y)$, $LSC^{\mathcal{A}}(X \rightrightarrows Y)$, $USC^{\mathcal{A}}(X \rightrightarrows Y)$ and $USC^{\mathcal{K}}(X \rightrightarrows Y)$ are respectively generated by the open sets

$$\begin{aligned}
\sigma_<^{\mathcal{MO}} &:= \{\{F \in LSC^{\mathcal{O}} \mid \forall x \in \bar{I}, y \in \bar{J},\ y \in F(x)\} \mid I \in \beta_X,\ J \in \beta_Y\},\\
\sigma_<^{\mathcal{MA}} &:= \{\{F \in LSC^{\mathcal{A}} \mid \bar{I} \subset F^{-1}(J)\} \mid I \in \beta_X,\ J \in \beta_Y\},\\
\sigma_>^{\mathcal{MA}} &:= \{\{F \in USC^{\mathcal{A}} \mid \bar{I} \cap F^{-1}(\bar{J}) = \emptyset\} \mid I \in \beta_X,\ J \in \beta_Y\},\\
\sigma_>^{\mathcal{MK}} &:= \{\{F \in USC^{\mathcal{K}} \mid \bar{I} \subset F^{\Leftarrow}(J_1 \cup \cdots \cup J_k)\} \mid I \in \beta_X,\ J_1, \ldots, J_k \in \beta_Y\}.
\end{aligned} \tag{2}$$

For more information on multivalued functions, see [16].

2.3. *Semicontinuity and Limits*

Given a set-valued function $F\colon X \rightrightarrows \mathcal{P}(Y)$ where X is a topological space, we can consider the functions formed by taking limits as $x' \to x$. Since $\mathcal{P}(Y)$ is a lattice, we can define the following operators:

$$\begin{aligned} \liminf F\,(x) &:= \bigcup_{U \ni x} \bigcap_{x' \in U} F(x') = \{y \in Y \mid \exists U \ni x,\ \forall x' \in U,\ y \in F(x')\}, \\ \limsup F\,(x) &:= \bigcap_{U \ni x} \bigcup_{x' \in U} F(x') = \{y \in Y \mid \forall U \ni x,\ \exists x' \in U,\ y \in F(x')\}, \end{aligned} \tag{3}$$

where U ranges over open subsets of X. Note that the above definition is purely set-theoretic in Y. If Y is a topological space, we can additionally define versions of lim sup and lim inf which take open or closed values.

Definition 2.2. Let X and Y be topological spaces, and $F\colon X \to \mathcal{P}(Y)$. Define the topological-theoretic limits:

$$\begin{aligned} \liminf{}^{\mathcal{O}} F\,(x) &:= \{y \in Y \mid \exists V \ni y,\ \exists U \ni x,\ \forall x' \in U,\ V \subset F(x')\}, \\ \liminf{}^{\mathcal{O}} F\,(x) &:= \{y \in Y \mid \forall V \ni y,\ \exists U \ni x,\ \forall x' \in U,\ F(x') \cap V \neq \emptyset\}, \\ \limsup{}^{\mathcal{A}} F\,(x) &:= \{y \in Y \mid \forall V \ni y,\ \forall U \ni x,\ \exists x' \in U,\ F(x') \cap V \neq \emptyset\}, \end{aligned} \tag{4}$$

where U ranges over open subsets of X, and V over open subsets of Y.

It is fairly straightforward to show that

$$\begin{aligned} \liminf{}^{\mathcal{O}} F\,(x) &:= \bigcup_{U \ni x} \operatorname{int}\left(\bigcap_{x' \in U} F(x') \right), \\ \limsup{}^{\mathcal{A}} F\,(x) &:= \bigcap_{U \ni x} \operatorname{cl}\left(\bigcup_{x' \in U} F(x') \right). \end{aligned} \tag{5}$$

The following result summarises the properties of $\liminf^{\mathcal{O}}$, $\liminf^{\mathcal{A}}$ and $\limsup^{\mathcal{A}}$ which we need.

Theorem 2.3.

1. *If* $F\colon X \rightrightarrows Y$, *then* $\liminf^{\mathcal{O}} F \in LSC^{\mathcal{O}}$, $\liminf^{\mathcal{O}} F \subset F$ *and* $F \in LSC^{\mathcal{O}} \iff F = \liminf^{\mathcal{O}} F$. *Further, if* $F \subset G$, *then* $\liminf^{\mathcal{O}} F \subset \liminf^{\mathcal{O}} G$.
2. *If* $F\colon X \rightrightarrows Y$, *then* $\liminf^{\mathcal{A}}$ *is closed-valued. If* F *is closed-valued, then* $\liminf^{\mathcal{A}} F \subset F$ *and* $F \in LSC^{\mathcal{A}} \iff F = \liminf^{\mathcal{A}} F$. *Further, if* $F \subset G$, *then* $\liminf^{\mathcal{A}} F \subset \liminf^{\mathcal{A}} G$.
3. *If* $F\colon X \rightrightarrows Y$, *then* $\limsup^{\mathcal{A}} F \in USC^{\mathcal{A}}$, $F \subset \limsup^{\mathcal{A}} F \subset F$ *and* $F \in USC^{\mathcal{A}} \iff F = \limsup^{\mathcal{A}} F$. *Further, if* $F \subset G$, *then* $\limsup^{\mathcal{A}} F \subset \limsup^{\mathcal{A}} G$.

The following example shows that $\liminf^{\mathcal{A}} F$ need not be lower-semicontinuous.

Example 2.4. Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by $F(x) = \{x\}$ if $x \in \mathbb{Q}$ and $F(x) = \{0\}$ otherwise. Then $\liminf^{\mathcal{A}} F(0) = \{0\}$ and $\liminf^{\mathcal{A}} F(x) = \emptyset$ if $x \neq 0$, so the function $\liminf^{\mathcal{A}} f$ is not lower-semicontinuous.

We can use Theorem 2.3 to find optimal semicontinuous approximations to functions. If $F: X \rightrightarrows Y$, then $\liminf^{\mathcal{O}} F$ is lower-semicontinuous with open values, and $\liminf^{\mathcal{O}} F(x) \subset F(x)$ for all x. If $G: X \rightrightarrows Y$ is lower-semicontinuous with open values, and $G(x) \subset F(x)$ for all x, then $\liminf^{\mathcal{O}} G \subset \liminf^{\mathcal{O}} F$, so $G \subset \liminf^{\mathcal{O}} F$. Hence $\liminf^{\mathcal{O}} F$ is the optimal lower-semicontinuous open-valued under-approximation to F. Similarly, if $G: X \rightrightarrows Y$ is upper-semicontinuous with closed values, and $F \subset G$, then $\limsup^{\mathcal{A}} F \subset G$, so $\limsup^{\mathcal{A}} F$ is the optimal upper-semicontinuous closed-valued over-approximation to F.

We remark that if F is lower-semicontinuous and β is a base for X, then F is completely determined by the values of $\bigcap\{F(x) \mid x \in J\}$ for $J \in \beta$. Similarly, if F is upper-semicontinuous, then F is completely determined by the values of $\bigcup\{F(x) \mid x \in J\}$ for $J \in \beta$.

2.4. *Approximations of Multivalued Maps*

Let (X, τ) be a second-countable locally compact Hausdorff space. We are interested in the function spaces $LSC^{\mathcal{A}}(X \rightrightarrows X)$ and $USC^{\mathcal{K}}(X \rightrightarrows X)$, and approximations in these spaces.

Choose a base β_1 for (X, τ) such that for all $I, J \in \beta_1$, then $I \cap J \in \beta_1$, and that if $I_0, I_1 \in \beta_1$ and $\bar{I}_1 \subset \bar{I}_0$, then either $I_1 = I_0$, or there exists $I_2 \in \beta_1$ such that $I_1 \cap I_2 = \emptyset$ and $\bar{I}_0 = \bar{I}_1 \cup \bar{I}_2$.

Choose a base β_2 for (X, τ) such that for all $I, J \in \beta_2$, $I \cup J \in \beta_2$, and for all $I \in \beta_1$, $J \in \beta_2$, we have $I \cap J = \emptyset \iff \bar{I} \cap \bar{J} = \emptyset$.

A base for the topology $\tau_{<}^{\mathcal{MA}}$ on $LSC^{\mathcal{A}}$ is given by sets of the form

$$\left\{\{F \in LSC^{\mathcal{A}} \mid \bar{I}_i \subset F^{-1}(J_i)\} \mid I_i \in \beta_1,\ J_i \in \beta_2\right\}. \tag{6}$$

A base for the topology $\tau_{>}^{\mathcal{MK}}$ on $USC^{\mathcal{K}}$ is given by sets of the form

$$\left\{\{F \in USC^{\mathcal{K}} \mid \bar{I}_i \subset F^{\Leftarrow}(J_i)\} \mid I_i \in \beta_1,\ J_i \in \beta_2\right\}. \tag{7}$$

Given $I_i \in \beta_1$, $J_i \in \beta_2$, $i = 1, \ldots, m$ such that the I_i are disjoint, choose $\tilde{I}_i \in \mathcal{O}(X)$ and $\hat{J}_i \in \mathcal{A}$ such that $\bar{I}_i \subset \tilde{I}_i$ and $\hat{J}_i \subset J_i$. We can further choose the $\tilde{I}_i$ and $\hat{J}_i$ such that if $\bar{I}_i \cap J_j \neq \emptyset$, then there exists $x \in \tilde{I}_i \cap \hat{J}_j$ such that $x \notin \tilde{I}_k$ for $k \neq i$.

Additionally, let $\tilde{I}_0 = X$, and take some compact set $\hat{J}_0$ such that $\bigcup_{i=1}^{m} J_i \subset \hat{J}_0$ Then the function defined by $F(x) := \bigcup\{\hat{J}_i \mid x \in \tilde{I}_i, i = 1, \ldots, m\}$ is lower-semicontinuous, and $\bar{I}_i \subset F^{-1}(J_i)$ for all i. Then the function defined by $F(x) := \bigcap\{\hat{J}_i \mid x \in \tilde{I}_i,\ i = 0, \ldots, m\}$ is upper-semicontinuous, and $\bar{I}_i \subset F^{\Leftarrow}(J_i)$ for all i.

Lemma 2.5.

1. *Take* $F \in LSC^{\mathcal{A}}$ *as above. Then* $F^{-1}(\tilde{I}_i) = \bigcup\{\tilde{I}_k \mid \bar{I}_i \cap J_j \neq \emptyset\}$.
2. *Take* $F \in USC^{\mathcal{K}}$ *as above. Then* $F(\hat{J}_j) = \bigcup\{\hat{J}_i \mid \bar{I}_i \cap J_j \neq \emptyset\}$.
3. *Take* $F \in USC^{\mathcal{A}}$ *as above, let* $\mathcal{I} \subset \{0, \ldots, m\}$, *and let* $U = \bigcup\{\bar{I}_i \mid i \in \mathcal{I}\}$ *and* $\tilde{U} = \bigcup\{\tilde{I}_i \mid i \in \mathcal{I}\}$. *Then* $F^{\Leftarrow}(\tilde{U}) = \bigcup\{\tilde{I}_j \mid J_j \subset U\}$.

Proof. 1. By construction, $F^{-1}(\tilde{I}_i) = \bigcup\{\tilde{I}_j \mid \tilde{I}_i \cap \hat{J}_j \neq \emptyset\} = \bigcup\{\tilde{I}_k \mid \bar{I}_i \cap J_j \neq \emptyset\}$.

2. If $\bar{I}_i \cap J_j \neq \emptyset$, then there exists $x \in \tilde{I}_i \cap \hat{J}_j$ such that $x \notin \tilde{I}_k$ for $k \neq i$. Then $F(x) = \hat{J}_i$, and $F(x') \subset \hat{J}_i$ for all $x' \in \tilde{I}_i$.

3. By construction, $F^{\Leftarrow}(\tilde{U}) = \bigcup\{\tilde{I}_j \mid \hat{J}_j \subset \tilde{U}\} = \bigcup\{\tilde{I}_j \mid J_j \subset U\}$. □

3. Computable Analysis and Topology

Computable analysis deals with real numbers, continuous functions on real and Euclidean spaces and subsets of Euclidean spaces. We assume familiarity with the definitions of notation and representations given in [23]. All the results of [23] carry over from the Euclidean case in a straightforward way, so we do not present proofs for the more general case here.

We take Σ to be a finite alphabet, and assume we have a tupling operation $\langle\cdot\rangle$ on Σ^*. We write $w \lhd p$ if $p = \langle w_1, w_2, \ldots\rangle$ and $w = w_i$ for some i.

We say a function $\eta: \Sigma^\omega \times \cdots \times \Sigma^\omega \to \Sigma^\omega$ is *computable* if there exists a Turing machine $\mathcal{M}$ which, on input $(p_1, \ldots, p_k)$, computes forever, writing the infinite sequence $p_0 = \eta(p_1, \ldots, p_k)$ to its output tape.

A *computable topological space* is a tuple (M, τ, σ, ν) such that X is a set, τ is a topology on X, σ is a countable sub-base for τ, and $\nu: \subset \Sigma^* \to \sigma$ is a partial surjective function giving a *notation* for σ. The *standard representation* of (M, τ, σ, ν) is the partial surjective function $\delta: \subset \Sigma^\omega \to X$ such that

$$\delta(p) = x \quad :\Longleftrightarrow \quad \{\nu(w) \mid w \lhd p\} = \{J \in \sigma \mid x \in J\}. \tag{8}$$

If $\delta_0, \ldots, \delta_k$ are representations $\delta_i: \subset \Sigma^\omega \to X_i$, then we say that $f: X_1 \times \cdots \times X_k \to X_0$ is $(\delta_1, \ldots, \delta_k; \delta_0)$-computable if there exists a computable function $\eta: \subset \Sigma^\omega \times \cdots \times \Sigma^\omega$ such that $f(\delta_1(y_1), \ldots, \delta_k(y_k)) = \delta_0(\eta(y_1, \ldots, y_k))$ whenever $y_i \in \operatorname{dom}(\delta_i)$ for all $i = 1, \ldots, k$.

The fundamental theorem of computable topology is that any computable function is continuous.

Theorem 3.1. *For $i = 0, \ldots, k$ let $\mathbf{S}_i = (M_i, \tau_i, \sigma_i, \nu_i)$ be a computable topological space, and let δ_i be the standard representation of $\mathbf{S}_i$. Then every $(\delta_1, \ldots, \delta_k; \delta_0)$-computable function $f: M_1 \times \cdots \times M_k \to M_0$ is $(\tau_1, \ldots, \tau_n; \tau_0)$-continuous.*

3.1. *Representations of Sets and Maps*

We now define representations of open, closed and compact sets, and of semicontinuous maps with closed and compact values. There are representations $\theta_<$ of $\mathcal{O}$, $\psi_<$ and $\psi_>$ of $\mathcal{A}$, and $\kappa_>$ of $\mathcal{K}$ defined as follows:

$$\begin{aligned}
\theta_<(p) = U \quad &:\Longleftrightarrow \quad \{\nu(w): w \lhd p\} = \{J \in \beta: \bar{J} \subset U\},\\
\psi_<(p) = A \quad &:\Longleftrightarrow \quad \{\nu(w): w \lhd p\} = \{J \in \beta: A \cap J \neq \emptyset\},\\
\psi_>(p) = A \quad &:\Longleftrightarrow \quad \{\nu(w): w \lhd p\} = \{J \in \beta: A \cap \bar{J} = \emptyset\},\\
\kappa_>(p) = C \quad &:\Longleftrightarrow \quad \{(\nu(w_1), \ldots, \nu(w_k)): \langle w_1, \ldots, w_k\rangle \lhd p\}\\
&\qquad\qquad = \{(J_1, \ldots, J_k) \subset \beta: C \subset \textstyle\bigcup_{i=1}^k J_i\}.
\end{aligned} \tag{9}$$

There are representations $\mu_<^\theta$ of $LSC_<^{\mathcal{O}}$, $\mu_<^\psi$ of $LSC_<^{\mathcal{A}}$, $\mu_>^\psi$ of $USC_>^{\mathcal{A}}$ and $\mu_>^\kappa$ of $USC_>^{\mathcal{K}}$ defined by

$$\begin{aligned}
\mu_<^{\mathcal{O}}(p)=F\in LSC_{\mathcal{O}} \;:\Longleftrightarrow\;& \{(\nu_X(v),\nu_Y(w)) : \langle v,w\rangle \lhd p\}\\
&= \{(I,J)\in\beta_X\times\beta_Y : \forall x\in\bar{I},\ y\in\bar{J},\ F(x)\ni y\},\\
\mu_<^{\mathcal{A}}(p)=F\in LSC_{\mathcal{A}} \;:\Longleftrightarrow\;& \{(\nu_X(v),\nu_Y(w)) : \langle v,w\rangle \lhd p\}\\
&= \{(I,J)\in\beta_X\times\beta_Y : \bar{I}\subset F^{-1}(J)\},\\
\mu_>^{\mathcal{A}}(p)=F\in USC_{\mathcal{A}} \;:\Longleftrightarrow\;& \{(\nu_X(v),\nu_Y(w)) : \langle v,w\rangle \lhd p\}\\
&= \{(I,J)\in\beta_X\times\beta_Y : \bar{I}\cap F^{-1}(\bar{J})=\emptyset\}\\
\mu_>^{\mathcal{K}}(p)=F\in USC_{\mathcal{K}} \;:\Longleftrightarrow\;& \{(\nu_X(v),\nu_Y(w_1),\ldots,\nu_Y(w_k)) : \langle v,w_1,\ldots,w_k\rangle \lhd p\}\\
&= \left\{(I,J_1,\ldots,J_k) : \bar{I}\subset F^{\Leftarrow}\left(\bigcup_{i=1}^{k} J_i\right)\right\}.
\end{aligned} \tag{10}$$

The representations $\theta_<$, $\psi_<$, $\psi_>$ and $\kappa_>$ are equivalent to the standard representations for the topologies $\tau_<^{\mathcal{O}}$, $\tau_<^{\mathcal{A}}$, $\tau_>^{\mathcal{A}}$ and $\tau_>^{\mathcal{K}}$, respectively. The representations $\mu_<^\theta$, $\mu_<^\psi$, $\mu_>^\psi$ and $\mu_>^\kappa$ are equivalent to the standard representations for the topologies $\tau_<^{\mathcal{MO}}$, $\tau_<^{\mathcal{MA}}$, $\tau_>^{\mathcal{MA}}$ and $\tau_>^{\mathcal{MK}}$, respectively.

3.2. *Computable Operations on Sets and Maps*

To prove computability of system-theoretic operators, we use the computability of important primitive operators on sets and multivalued maps. We first show that most important set-theoretic operators are computable.

Theorem 3.2.

1. *Closure* $U\mapsto \mathrm{cl}(U)$ *is* $(\theta_<;\psi_<)$*-computable.*
2. *Union* $(U,V)\mapsto U\cup V$ *is* $(\theta_<,\theta_<;\theta_<)$*-computable,* $(A,B)\mapsto A\cap B$ *is* $(\psi_<,\psi_<;\psi_<)$*-computable and* $(\psi_>,\psi_>;\psi_>)$*-computable, and* $(C,D)\mapsto C\cup D$ *is* $(\kappa_>,\kappa_>;\kappa_>)$*-computable.*
3. *Intersection* $(A,B)\mapsto A\cap B$ *is* $(\psi_>,\psi_>;\psi_>)$*-computable, and* $(A,C)\mapsto A\cap C$ *is* $(\psi_>,\kappa_>;\kappa_>)$*-computable.*
4. *Closed intersection* $(A,U)\mapsto \mathrm{cl}(A\cap U)$ *is* $(\psi_<,\theta_<;\psi_<)$*-computable.*
5. *Set difference* $(U,A)\mapsto U\setminus A$ *is* $(\theta_<,\psi_>;\theta_<)$*-computable, and* $(A,U)\mapsto A\setminus U$ *is* $(\psi_>,\theta_<;\psi_>)$*-computable.*

Note that intersection $(A,B)\mapsto A\cap B$ is not $(\psi_<,\psi_<;\psi_<)$-computable.

We next show that certain limits of sets are computable. Each of these limiting operations is closely connected with convergence in the respective topology. We topologise the infinite product space $M_1\times M_2\times\cdots$ using the product topology, and take as representation $\delta\langle p_1,p_2,\ldots\rangle=\delta_1(p_1),\delta_2(p_2),\ldots$, where $\langle p_1,p_2,\ldots\rangle$ is the tupling operation of countably many infinite sequences defined using the Gödel ordering.

Theorem 3.3.

1. *Let* $(U_1,U_2,\ldots)$ *be a sequence of open sets such that* $U_i\subset U_j$ *whenever* $i<j$*. Then* $\lim_{i\to\infty}U_i$ *exists in* $\tau^{\mathcal{O}}$ *and the operator* $(U_1,U_2,\ldots)\mapsto\lim_{i\to\infty}U_i$ *is* $(\theta_<,\theta_<,\ldots;\theta_<)$*-computable.*

2. *Let* $(A_1, A_2, \ldots)$ *be a sequence of closed sets such that* $A_i \subset N_{2^{-i}}(A_j)$ *whenever* $i < j$. *Then* $\lim_{i\to\infty} A_i$ *exists in* $\tau^{\mathcal{A}}$ *and the operator* $(A_1, A_2, \ldots) \mapsto \lim_{i\to\infty} A_i$ *is* $(\psi_<, \psi_<, \ldots; \psi_<)$*-computable.*
3. *Let* $(A_1, A_2, \ldots)$ *be a sequence of closed sets such that* $A_j \subset A_i$ *whenever* $i < j$. *Then* $\lim_{i\to\infty} A_i$ *exists in* $\tau^{\mathcal{A}}$ *and the operator* $(A_1, A_2, \ldots) \mapsto \lim_{i\to\infty} A_i$ *is* $(\psi_>, \psi_>, \ldots; \psi_>)$*-computable.*
4. *Let* $(C_1, C_2, \ldots)$ *be a sequence of compact sets such that* $C_j \subset C_i$ *whenever* $i < j$. *Then* $\lim_{i\to\infty} C_i$ *exists in* $\tau^{\mathcal{K}}$ *and the operator* $(C_1, C_2, \ldots) \mapsto \lim_{i\to\infty} C_i$ *is* $(\kappa_<, \kappa_<, \ldots; \kappa_<)$*-computable.*

We now consider images and preimages of sets under semicontinuous maps. The following theorem is proved in [6], and we provide a sketch of the proof of Theorem 3.5. Certain strong preimages are also computable, but we do not need these here.

Theorem 3.4. *Let* X *and* Y *be computable Hausdorff spaces, let* $F: X \rightrightarrows Y$ *be a multivalued function, let* $U \subset X$ *be an open set, let* $A \subset X$ *be a closed set and let* $C \subset X$ *be a compact set.*

1. *The operator* $(F, U \mapsto F(U)$ *is* $(\mu_<^{\mathcal{O}}, \theta_<; \theta_<)$*-computable for* $F \in LSC^{\mathcal{O}}(X \rightrightarrows Y)$.
2. *The operator* $(F, A) \mapsto \mathrm{cl}(F(A))$ *is* $(\mu_<^{\mathcal{A}}, \psi_<; \psi_<)$*-computable for* $F \in LSC^{\mathcal{A}}(X \rightrightarrows Y)$.
3. *The operator* $(F, C) \mapsto F(C)$ *is* $(\mu_>^{\mathcal{A}}, \kappa_>; \psi_>)$*-computable for* $F \in USC^{\mathcal{A}}(X \rightrightarrows Y)$.
4. *The operator* $(F, C) \mapsto F(C)$ *is* $(\mu_>^{\mathcal{K}}, \kappa_>; \kappa_>)$*-computable for* $F \in USC^{\mathcal{K}}(X \rightrightarrows Y)$.

Note that the operator $(F, A) \mapsto \mathrm{cl}(F(A))$ is not $(\mu^{\mathcal{K}}, \psi; \psi_>)$-computable for $F \in USC^{\mathcal{K}}$, since it is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{A}}; \tau_>^{\mathcal{A}})$-continuous.

Theorem 3.5. *Let* X *and* Y *be computable Hausdorff spaces, let* $F: X \rightrightarrows Y$ *be a multivalued function and let* U *be an open set.*

1. *The operator* $(F, U) \mapsto F^{-1}(U)$ *is* $(\mu_<^{\mathcal{A}}, \theta_<; \theta_<)$*-computable* $F \in LSC^{\mathcal{A}}(X \rightrightarrows Y)$.
2. *The operator* $(F, A) \mapsto F^{-1}(A)$ *is* $(\mu_>^{\mathcal{K}}, \psi_>; \psi_>)$*-computable* $F \in USC^{\mathcal{K}}(X \rightrightarrows Y)$.
3. *The operator* $(F, C) \mapsto F^{-1}(C)$ *is* $(\mu_>^{\mathcal{A}}, \kappa_>; \psi_>)$*-computable* $F \in USC^{\mathcal{A}}(X \rightrightarrows Y)$.

Proof. 1. $\bar{L} \subset F^{-1}(U)$ if, and only if, there exist $\bar{I}_1, \ldots, \bar{I}_m, J_1, \ldots, J_m, K_1, \ldots, K_n$ such that $\bar{L}_i \subset \bigcup_{i=1}^m I_i$, $\bar{I}_i \subset F^{-1}(J_i)$ for $i = 1, \ldots, m$, and $\bar{J}_i \subset \bigcup_{j=1}^n K_j$ for $i = 1, \ldots, m$.

2. $\bar{I} \cap F^{-1}(A) = \emptyset$ if, and only if, there exist $J_1, \ldots, J_k$ such that $F(\bar{I}) \subset \bigcup_{i=1}^k J_i$ and $\bar{J}_i \cap A = \emptyset$ for all $i = 1, \ldots, k$.

3. $\bar{I} \cap F^{-1}(C) = \emptyset$ if, and only if, there exist $J_1, \ldots, J_k$ such that $C \subset \bigcup_{i=1}^k J_i$ and $F(\bar{I}) \cap \bar{J} = \emptyset$ for all $i = 1, \ldots, k$. □

4. Reachability and Invariance Problems

We now apply the material developed in Section 2.2 to the study of the reachability problem for semicontinuous systems. We first define the reachable, closed-reachable and chain-reachable sets, and give an alternative formulation of the chain reachable set. We then prove some straightforward results on computability of countable unions and intersections, and use these to prove the main results on reachability. Finally, we discuss *closure-interior systems*, which have inner as well as outer approximations, and show that the computability results extend to these systems as well.

Viable and invariant sets are also important system properties. Recall that a set A is *viable* for a system F if, for every point x of A, there is an orbit through x remaining in A, and *invariant* if every orbit starting in A remains in A. A viable set may also be described as *control-invariant*, and an invariant set as *perturbation invariant*. See [3] for a detailed exposition of viability theory.

4.1. *Computability of Reachable Sets*

Definition 4.1 (Reachability). Let $F\colon X \rightrightarrows X$ be a multivalued map, and $X_0 \subset X$. Then the *reachable set* of F from X_0 is

$$\begin{aligned}\operatorname{Reach}(F, X_0) &\\ &:= \{x \in X \mid \exists\, x_0, \ldots, x_n \text{ s.t. } x_0 \in X_0,\ x_n = x,\ \text{and } \forall\, i,\ x_{i+1} \in F(x_i)\} \\ &= \bigcup_{n=0}^{\infty} F^n(X_0). \qquad (11)\end{aligned}$$

If F has open values and X_0 is open, then $\operatorname{Reach}(F, X_0)$ is open. However, even if F is continuous with compact values, and X_0 is compact, the reachable set need not be closed, so we take its closure, and define the *closed reachable set* as

$$\operatorname{clReach}(F, X_0) := \operatorname{cl}(\operatorname{Reach}(F, X_0)). \qquad (12)$$

The following theorem [6] shows that the closed reachable set is lower-semi-computable:

Theorem 4.2.

1. $(F, U) \mapsto \operatorname{Reach}(F, U)$ *is* $(\mu^{\theta}_{<}, \theta_{<}; \theta_{<})$*-computable.*
2. $(F, A) \mapsto \operatorname{clReach}(F, A)$ *is* $(\mu^{\psi}_{<}, \psi_{<}; \psi_{<})$*-computable.*

Unfortunately, $(F, C) \mapsto \operatorname{clReach}(F, C)$ is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}; \tau^{\mathcal{A}}_{>})$-continuous, so is not $(\mu^{\kappa}, \kappa; \psi_{>})$-computable. To find an upper-semicontinuous over-approximation to the reachable set, we introduce the concept of ε-chains as considered by Conley [8].

Definition 4.3. $F\colon X \rightrightarrows X$ and $\varepsilon > 0$. A sequence $x_0, \ldots, x_n$ is an ε*-chain* for F if there exist points $y_1, \ldots, y_n \in X$ such that $y_{i+1} \in F(x_i)$ and $d(x_{i+1}, y_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. The *chain reachable set* of F from X_0 is defined

$$\begin{aligned}\operatorname{ChainReach}(F, X_0) &\\ &:= \{x \in X \mid \forall \varepsilon > 0,\ \exists\ \varepsilon\text{-chain } x_0, \ldots x_n \text{ with } x_0 \in X_0 \text{ and } x_n = x\}. \qquad (13)\end{aligned}$$

Clearly, $\mathrm{Reach}(F, C) \subset \mathrm{ChainReach}(F, C)$. For our purposes, however, it is more convenient to use the following metric-free characterisation:

Theorem 4.4. *Let $F \in USC^{\mathcal{K}}$ and C a compact set. Suppose* $\mathrm{ChainReach}(F, C)$ *is compact. Then*

$$\mathrm{ChainReach}(F, C) = \bigcap\{U \in \mathcal{O}(X) \mid C \subset U \text{ and } \mathrm{cl}(U) \subset F^{\Leftarrow}(U)\}. \tag{14}$$

The following result [6] shows that the chair-reachable set is upper-semicomputable.

Theorem 4.5. *If* $\mathrm{ChainReach}(F, C)$ *is compact, then* $\mathrm{ChainReach}(F, C)$ *is* $(\mu^{\kappa}_{>}, \kappa_{>}; \kappa_{>})$*-computable.*

If $\mathrm{ChainReach}(F, C)$ is not compact, then ChainReach need not be $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}; \tau^{\mathcal{A}}_{>})$-continuous at (F, C), as is the case in Example 4.7 of [6]. The difficulty is that it is impossible to have considered the entire chain-reachable set at any finite stage in the computation, and hence it is impossible to prove that any point is unreachable.

By Theorem 3.1, any $(\mu^{\kappa}_{>}, \kappa_{>}; \kappa_{>})$-computable function is $(\tau^{\mathcal{MK}}_{>}, \tau^{\mathcal{K}}_{>}; \tau^{\mathcal{K}}_{>})$-continuous. It therefore remains to show that ChainReach is the best-possible upper-semicontinuous over-approximation to $\mathrm{Reach}(F, C)$.

Theorem 4.6. *Suppose* $\mathrm{ChainReach}(F, C)$ *is compact. Then* $\mathrm{ChainReach}(F, C) = [\limsup^{\mathcal{A}} \mathrm{Reach}](F, C)$

Proof. Let N_F be a basic open neighbourhood of F defined by $\tilde{F}(\bar{I}_i) \subset J_i$ for $i = 1, \ldots, m-1$ for all $\tilde{F} \in N_F$. Let N_C be a basic open neighbourhood of C defined by $\tilde{C} \subset J_0$ for all $\tilde{C} \in N_C$, and take $\bar{I}_0 = \emptyset$. Take $\bar{I}_m$ so that $\{I_1, \ldots, I_m\}$ is a topological partition of X, and J_m such that $\bar{I}_i \cup \bar{J}_i \subset J_m$ for $i = 0, \ldots, m-1$.

Define sets $\mathcal{I}_k$ as follows: Let $\mathcal{I}_0 = \{0\}$, and define $\mathcal{I}_k$ recursively by $\mathcal{I}_k = \mathcal{I}_{k-1} \cup \{i \in 0, \ldots, m \mid \exists j \in \mathcal{I}_{k-1}, \bar{I}_i \cap J_j \cap \neq \emptyset\}$. Since the $\mathcal{I}_k$ are an increasing sequence of subsets of $\{0, 1, \ldots, m\}$, the sets eventually limit on some set $\mathcal{I}_\infty$, with the property that $\forall j \in \mathcal{I}_\infty$, $J_j \subset \bigcup\{\bar{I}_i \mid i \in \mathcal{I}\}$.

Suppose $m \notin \mathcal{I}_\infty$, and define $V = \bigcup\{J_j \mid j \in \mathcal{I}\}$. Then $C \in V$, and $\mathrm{cl}(V) \subset \bigcup\{\bar{I}_i \mid \bar{I}_i \cap V \neq \emptyset\} \subset \bigcup\{\bar{I}_i \mid i \in \mathcal{I}\} \subset F^{\Leftarrow}(V)$. Therefore $\mathrm{ChainReach}(F, C) \subset V$.

Now construct upper-semicontinuous $\hat{F}$ as in Section 2.4, and take $\hat{C} = \hat{J}_0$. Then $\hat{F} \in N_F$ and $\hat{C} \in N_C$. Further, it is easy to see that $\mathrm{Reach}(\hat{F}, \hat{C}) = \bigcup\{\hat{J}_i \mid i \in \mathcal{I}_\infty\}$. Hence $V = \bigcup\{\mathrm{Reach}(\hat{F}, \hat{C}) \mid \hat{F} \in N_F, \hat{C} \in N_C\}$.

We therefore have $\mathrm{ChainReach}(F, C) \subset V$, and $V = \bigcup\{\mathrm{Reach}(\hat{F}, \hat{C}) \mid \hat{F} \in N_F,\ \hat{C} \in N_C\}$. Hence $\mathrm{ChainReach}(F, C) \subset \limsup^{\mathcal{A}} \mathrm{Reach}\ (F, C)$. □

4.2. *Computation of Viability Kernels*

We first consider the computation of the maximal viable subset of a given set.

Definition 4.7. The *viability kernel* of B under F is

$$\begin{aligned}\mathrm{Viab}(F, B) &:= \{x \mid \exists\, x_0, x_1, \ldots \text{ s.t. } x = x_0, \text{ and } \forall i,\ x_{i+1} \in F(x_i) \text{ and } x_i \in B\} \\ &= \bigcap_{n=0}^{\infty} F^{-n}(B).\end{aligned} \tag{15}$$

It was shown by Saint-Pierre [20] that if C is compact, the viability kernel varies upper-semicontinuously in (F, C), and an algorithm to compute it was given. The viability kernel is also upper-semicomputable in the framework of computable analysis.

Theorem 4.8.

1. $(F, A) \mapsto \mathrm{Viab}(F, A)$ *is* $(\mu_>^\kappa, \psi_>; \psi_>)$*-computable.*
2. $(F, C) \mapsto \mathrm{Viab}(F, C)$ *is* $(\mu_>^\psi, \kappa_>; \kappa_>)$*-computable.*

Proof. 1. Since $(F, A) \mapsto F^{-1}(A)$ is $(\mu_>^\kappa, \psi_>; \psi_>)$, we can compute a $\psi_>$-name of $F^{-1}(A)$ from a $\psi_>$-name of A, and hence recursively compute a $\psi_>$-name of $F^{-n}(A)$ for all $n \in \mathbb{Z}^+$. The result follows since the sequence $\bigcap_{i=1}^n F^{-i}(A)$ is a decreasing sequence of $\psi_>$-computable closed sets, so the limit $\bigcap_{i=1}^\infty F^{-i}(A)$ is also $\psi_>$-computable.

2. Let $C_0 = C$, and define $C_{n+1} = C_n \cap F^{-1}(C_n)$. Then we can compute a $\kappa_>$-name of C_n for all n, since $(F, C) \mapsto F^{-1}(C)$ is $(\mu_>^\psi, \kappa_>; \psi_>)$-computable, and $(C, A) \mapsto C \cap A$ is $(\kappa_>, \psi_>; \kappa_>)$-computable. Then result follows since the C_n is a decreasing sequence of $\kappa_>$-computable compact sets, and $\mathrm{Viab}(F, C) = \lim_{n\to\infty} C_n$. □

Unfortunately, it is not possible to compute a good lower-approximation to $\mathrm{Viab}(F, C)$ for a compact set C. The operator $(F, C) \mapsto \mathrm{Viab}(F, C)$ is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{K}}; \tau_<^{\mathcal{A}})$-continuous, so is not $(\mu^\kappa, \kappa; \psi_<)$-computable, as the following example shows.

Example 4.9. Let $F(x) = 2x$ and $C = [0, 1]$. We can take approximations C_n to C by finite sets of rational points, and (lower or upper) semicontinuous approximations F_n to F mapping rational points to irrational points. Then $F_n(C_n) \cap C_n = \emptyset$ for all n, so $\mathrm{Viab}(F_n, C_n) = \emptyset$. Hence we have $\liminf_{(F',C')\to(F,C)} \mathrm{Viab}(F, C) = \emptyset$.

This example can be used to prove the following result.

Theorem 4.10. *For all* $F \in C^{\mathcal{K}}(X \rightrightarrows X)$, $C \in \mathcal{K}(X)$, $[\liminf^{\mathcal{A}} \mathrm{Viab}](F, C) = \emptyset$, *taking topology* $\tau^{\mathcal{MK}}$ *on* $C^{\mathcal{K}}(X \rightrightarrows X)$ *and* $\tau^{\mathcal{K}}$ *on* $\mathcal{K}(X)$.

Proof. Let Ξ be a dense subset of X, and approximate C by finite subsets C_n of Ξ. We can then always approximate F in $(C^{\mathcal{K}}(X \rightrightarrows X), \tau^{\mathcal{MK}})$ by a sequence F_n such that $F_n(x) \cap C_n = \emptyset$ for all $x \in C_n$. Hence $\liminf^{\mathcal{A}} \mathrm{Viab} = \emptyset$. □

The following example shows that the viability kernel may depend continuously on the system.

Example 4.11. Consider $F \in C(\mathbb{R} \rightrightarrows \mathbb{R})$ given by $F(x) = \{2x\}$, and $C = [-1, 1]$. Then, clearly, $\mathrm{Viab}(F, C) = \{0\}$. Further, $\mathrm{Viab}(F, C) \neq \emptyset$ for any continuous perturbation of F in $C(\mathbb{R} \rightrightarrows \mathbb{R})$.

Recall that a set A is viable if $A \subset F^{-1}(A)$. We say that A is *robustly viable* if $\mathrm{cl}(A) \subset F^{-1}(\mathrm{int}(A))$.

Definition 4.12. The *robust viability kernel* of B is

$$\mathrm{RobustViab}(F, B) := \bigcup\{C \in \mathcal{K} \mid C \subset \mathrm{int}(B) \cap F^{-1}(\mathrm{int}(C))\}. \tag{16}$$

If F is lower-semicontinuous, then $F^{-1}(V)$ is open whenever V is open, and it is easy to see that the robust viability kernel is open. Using Theorem 3.5, we can show it is also computable.

Theorem 4.13. *The operator* $(F, U) \mapsto \mathrm{RobustViab}(F, U)$ *is* $(\mu_<^{\psi}, \theta_<; \theta_<)$*-computable.*

The following result shows that the robust viability kernel is the optimal lower-semicomputable under-approximation to the viability kernel.

Theorem 4.14. $\liminf^{\mathcal{O}} \mathrm{Viab} = \mathrm{RobustViab}$ *on* $LSC^{\mathcal{A}} \times \mathcal{O}_<$.

Proof. Let N_F be a basic open set in $\tau_<^{\mathcal{MA}}$ given by $N_F = \{\tilde{F} \in LSC^{\mathcal{A}} \mid \bar{I}_i \subset \tilde{F}^{-1}(J_i)$ for $i = 1, \ldots, m\}$. Let N_U be a basic open set in $\tau_<^{\mathcal{O}}$ given by $N_U = \{\tilde{U} \in \mathcal{O} \mid \bar{I}_i \subset \tilde{U}\ \forall i \in \mathcal{I}_0\}$. Take $J_0 = X$.

We now attempt to compute a set C such that C is viable for all $\tilde{F} \in N_F$, $\tilde{U} \in N_U$. Define $\mathcal{I}_k$ recursively by $\mathcal{I}_k := \{j \in \mathcal{I}_{k-1} \mid \bar{I}_i \cap J_j \neq \emptyset$ for some $i \in \mathcal{I}_{k-1}\}$. The sets $\mathcal{I}_k$ are decreasing finite sets, so eventually stabilise to a set $\mathcal{I}_\infty$, with the property that if $J_j \cap \bar{I}_i \neq \emptyset$ for some $i \in \mathcal{I}_\infty$, then $j \in \mathcal{I}_\infty$.

Let $C = \bigcup\{\bar{I}_i \mid i \in \mathcal{I}_\infty\}$, $V = \bigcup\{J_i \mid i \in \mathcal{I}_\infty$, and let $F \in N_F$. Then $\bar{I}_i \subset J_i$, so $C \subset F^{-1}(V)$. Further, if $j \in \mathcal{I}_\infty$ and $\bar{I}_i \cap J_j \neq \emptyset$, then $i \in \mathcal{I}_\infty$, so $V \subset C$. Hence $C \subset F^{-1}(\mathrm{int}(C))$, and by construction, $C \subset \bigcup\{\bar{I}_i \mid i \in \mathcal{I}_0\}$, so $C \subset U$ for all $U \in N_U$. Therefore $C \subset \mathrm{RobustViab}(F, U)$ for all $F \in N_F$ and $U \in N_U$.

By the construction in Section 2.4, we can construct lower-semicontinuous $\tilde{F}$ such that $\tilde{U}_k \cap F^{-1}(\tilde{U}_k) = \tilde{U}_{k+1}$, where $\tilde{U}_k = \bigcup\{\tilde{I}_i \mid i \in \mathcal{I}_k\}$. Then $\mathrm{Viab}(\tilde{F}, \tilde{U}_0) = \tilde{U}_\infty$, and for the sets $\tilde{I}_i$ sufficiently close to $\bar{I}_i$, we have $\bigcap\{\mathrm{Viab}(\tilde{F}, \tilde{U}) \mid \tilde{F} \in N_F,\ \tilde{U} \in N_U\} = C$.

We therefore have $C \subset \mathrm{RobustViab}(F, U)$ and $C = \bigcap\{\mathrm{Viab}(\tilde{F}, \tilde{U}) \mid \tilde{F} \in N_F,\ \tilde{U} \in N_U\}$. Hence $\liminf^{\mathcal{O}} \mathrm{Viab}\ (F, U) \subset \mathrm{RobustViab}(F, U)$, but since RobustViab is lower-semicontinuous and $\mathrm{RobustViab}(F, U) \subset \mathrm{Viab}(F, U)$ for all U, we must have equality $\liminf^{\mathcal{O}} \mathrm{Viab}\ (F, U) = \mathrm{RobustViab}(F, U)$. □

4.3. *Computation of Invariance Kernels*

We now consider computability of the maximal invariant subset of a given set.

Definition 4.15. The *invariance kernel* of B under F is

$$\begin{aligned}\mathrm{Inv}(F, B) &:= \{x \mid \forall x_0, x_1, \ldots \text{ s.t. } x_0 = x \text{ and } x_{i+1} \in F(x_i), x_i \in B \; \forall i\} \\ &= X \setminus \bigcup_{n=0}^{\infty} F^{-n}(X \setminus B).\end{aligned} \tag{17}$$

We obtain the following result on computability of the invariance kernel:

Theorem 4.16.

1. $(F, A) \mapsto \mathrm{Inv}(F, A)$ *is* $(\mu_<^\psi, \psi_>; \psi_>)$*-computable.*
2. $(F, C) \mapsto \mathrm{Inv}(F, C)$ *is* $(\mu_<^\psi, \kappa_>; \kappa_>)$*-computable.*

Proof. 1. Let $U = X \setminus A$, which is $\theta_<$-computable. By Theorem 3.5, $F^{-n}(U)$ is $\theta_<$-computable for all n. By Theorems 3.2 and 3.3, $\bigcup_{n=0}^{\infty} F^{-n}(U)$ is $\theta_<$-computable. Hence $\mathrm{Inv}(F, A) = X \setminus \bigcup_{n=0}^{\infty} F^{-n}(X \setminus A)$ is $\psi_>$-computable.

2. By (2), $\mathrm{Inv}(F, C)$ is $\psi_>$-computable. Since $\mathrm{Inv}(F, C) = \mathrm{Inv}(F, C) \cap C$, we immediately see that $\mathrm{Inv}(F, C)$ is $\kappa_>$-computable by Theorem 3.2. □

Notice that we can compute an *upper* approximation to $\mathrm{Inv}(F, C)$ using a *lower* approximation to F.

Unfortunately, $(F, C) \mapsto \mathrm{Inv}(F, C)$ is not $(\tau^{\mathcal{MK}}, \tau^{\mathcal{A}}; \tau_<^{\mathcal{A}})$-continuous, so is not $(\mu^\kappa, \kappa; \psi_<)$-computable. Indeed, just as in the case of the viability kernel, $\liminf_{(F',C') \to (F,C)} \mathrm{Inv}(F', C') = \emptyset$ for all (F, C). To obtain lower approximations to the invariance kernel, we consider robust invariance. Recall that a set A is invariant if $F(A) \subset A$, or equivalently, $A \subset F^{\Leftarrow}(A)$. We say that A is *robustly invariant* if $\mathrm{cl}(A) \subset F^{\Leftarrow}(\mathrm{int}(A))$.

Definition 4.17. The *robust invariance kernel* of B is

$$\mathrm{RobustInv}(F, B) := \bigcup \{C \in \mathcal{K} \mid C \subset \mathrm{int}(B) \cap F^{\Leftarrow}(\mathrm{int}(C))\}.$$

If F is upper-semicontinuous, then $F^{\Leftarrow}(V)$ is open whenever V is open, and it is easy to see that the robust invariance kernel is open. We have the following computability result:

Theorem 4.18. *The operator* $(F, U) \mapsto \mathrm{RobustInv}(F, U)$ *is* $(\mu_>^\kappa, \theta_<; \theta_<)$*-computable.*

The following result shows that the robust invariance kernel is the optimal lower-semicomputable under-approximation to the invariance kernel.

Theorem 4.19. $\liminf^{\mathcal{O}} \mathrm{Inv} = \mathrm{RobustInv}$ *on* $USC^{\mathcal{K}} \times \mathcal{O}_<$.

Proof. Let N_F and N_U be basic open neighbourhoods defined by $F(\bar{I}_i) \subset J_i$ for $i = 1, \ldots, n$ and $\bigcup\{\bar{I}_i \mid i \in \mathcal{I}_0\} \subset U$. Take $\bar{I}_m$ so that $\bar{I}_1, \ldots, \bar{I}_m$ is a topological partition of X, and J_m so that $\bar{I}_i, \bar{J}_i \subset J_m$ for $i = 1, \ldots, m-1$.

Define sets $\mathcal{I}_k$ and W_k by $\mathcal{I}_k = \{i \in \mathcal{I}_{k-1} \mid J_i \not\subset W_{k-1}\}$ and $W_k = \bigcup\{\bar{I}_i \mid i \in \mathcal{I}_k\}$. Let $\mathcal{I}_\infty$ be the limit of the $\mathcal{I}_k$, $W = W_\infty$, and $V = \bigcup\{J_i \mid i \in \mathcal{I}_\infty\}$. Then if $j \in \infty$, $J_j \in \bigcup\{\bar{I}_i \mid i \in \mathcal{I}_\infty\}$.

Then by construction, if $F \in N_F$, we have $\bar{I}_i \subset F^{\Leftarrow}(J_i)$ for all i, so $W \subset F^{\Leftarrow}(V)$. Further, since $J_j \subset W$ for all $j \in \mathcal{I}_\infty$, we have $V \subset W$, so $W \subset F^{\Leftarrow}(\text{int}(W))$. If $U \in N_F$, then $W \subset W_0 \subset U$, so $W \subset \text{RobustInv}(F, U)$.

By the construction in Section 2.4, we can construct upper-semicontinuous $\hat{F}$ such that $\tilde{U}_k \cap \hat{F}^{\Leftarrow}(\tilde{U}_k) = \tilde{U}_{k+1}$, where $\tilde{U}_k = \bigcup\{\tilde{I}_i \mid i \in \mathcal{I}_k\}$. Then $\text{Inv}(\hat{F}, \tilde{U}_0) = \tilde{U}_\infty$, and the sets $\tilde{I}_i$ sufficiently close to $\bar{I}_i$, we have $\bigcap\{\text{Viab}(\hat{F}, \tilde{U}) \mid \hat{F} \in N_F,\ \tilde{U} \in N_U\} = W$.

We therefore have $W \subset \text{RobustInv}(F, U)$ and $W = \bigcap\{\text{Inv}(\hat{F}, \tilde{U}) \mid \hat{F} \in N_F,\ \tilde{U} \in N_U\}$. Hence $\liminf^{\mathcal{O}} \text{Inv}\ (F, U) \subset \text{RobustInv}(F, U)$, but since RobustInv is lower-semicontinuous and $\text{RobustInv}(F, U) \subset \text{Inv}(F, U)$ for all U, we must have equality $\liminf^{\mathcal{O}} \text{Inv}\ (F, U) = \text{RobustInv}(F, U)$. □

5. Conclusions and Further Research

In this paper we have considered the computation of reachable, viable and invariant sets in the setting of computable analysis and topology. We have seen that the reachable set is lower-semicomputable, whereas viability and invariance kernels are upper-semicomputable. We have shown that the chain-reachable set is the best upper-semicomputable approximation to the reachable set, and that the robust viability and invariance kernels are the best lower-semicomputable approximations to the viable and invariance kernels. We have also seen that nontrivial semicomputable under-approximations to the viable and invariance kernels can only be computed for open sets, and not for closed sets with the lower topology The results in this paper complete the study of basic dynamical properties of multivalued maps begun in [6] and [7] by showing that the results obtained are optimal.

The methods used are to construct approximations to the sets of interest valid in some neighbourhood of the parameters. We show that the chain-reachable set is the limit-supremum of the reachable set, and the robust viability and invariance kernels are the limit-infimum of the viability and invariance kernels. We then use general properties of lim sup and lim inf to prove that the approximations obtained are optimal.

The methods used provide a general foundation to consider optimal computable approximations in other settings. Whenever a function is not continuous, we attempt to find a lower-semicontinuous under-approximation, and an upper-semicontinuous over-approximation. If these functional are computable, they provide the optimal computable approximation to the function of interest. Important uncomputable problems occur in fixed-point theory and nonlinear dynamics, such as the computation of invariant sets and topological entropy, and the computation of optimal controllers.

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